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LETTER TO THE EDITOR

Some new non-diagonal representations for density operators of optical fields

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Abstract. Some new non-diagonal representations for density operators of radiation in terms of coherent states are given. These involve only two integrations per mode. The weight functions are expressed in terms of coherence functions and of occupation number space matrix elements of the density operators.

When a compact form of the weight function in the diagonal representation does not exist, one of these new representations may become preferable. One such example is discussed.

The density operators of optical fields can always be expressed in the diagonal representation (Sudarshan 1963, Glauber 1963),

$$\rho = \int d^2\alpha P(\alpha)|\alpha\rangle\langle\alpha|, \tag{1}$$

where $|\alpha\rangle$ is the coherent state. The weight function $P(\alpha)$ can be expressed in terms of the occupation number space matrix elements, ρ_{mn}, of the density operator in the form (Sudarshan 1963),

$$P(|\alpha|\,e^{-i\theta}) = \sum_{m,n} [2\pi|\alpha|(m+n)!]^{-1}(m!n!)^{1/2}\rho_{mn}\exp[|\alpha|^2+i(m-n)\theta]\delta^{(m+n)}(|\alpha|), \tag{2}$$

where $\delta^{(m+n)}$ is the derivative of order $(m+n)$ of the Dirac delta, δ. Although equations (1) and (2) are quite general (see, eg Sudarshan 1963, Mehta and Sudarshan 1965, Klauder and Sudarshan 1968), it is desirable to have a compact form of the weight function $P(\alpha)$. For example, for chaotic optical field having $\bar{n}$ mean number of photons, the form,

$$P(\alpha) = (\pi\bar{n})^{-1}\exp(-\bar{n}^{-1}|\alpha|^2), \tag{3}$$

of the weight function $P(\alpha)$ is preferable to the form,

$$P(\alpha) = \sum_{n} [2\pi|\alpha|(2n)!]^{-1}n!(1+\bar{n})^{-n-1}\bar{n}^n\,e^{|\alpha|^2}\delta^{(2n)}(|\alpha|). \tag{4}$$

There have been attempts (Cahill 1969, Lu 1971, Chandra and Prakash 1973, Prakash *et al* 1974a) to modify equation (1) so that the weight functions may not have a highly singular form. The purpose of this letter is to report some new non-diagonal representations† which also involve only two integrations per mode. Calculations in

† The density operators can also be expressed in the well known non-diagonal R representation (Glauber 1963), $\rho = \int d^2\alpha\, d^2\beta\exp[-\frac{1}{2}(|\alpha|^2+|\beta|^2)]R(\alpha^*,\beta)|\alpha\rangle\langle\beta|$, but this representation involves four integrations. Calculations in this representation are, thus, more complicated than those in the diagonal representation.

these representations are, thus, as convenient as they are in the diagonal representation. For some optical fields, some of these representations may have compact convenient forms of the weight functions. If the weight function in the diagonal representation is not compact, these non-diagonal representations are then preferable. We shall later in this letter discuss one such example.

If we can find N identities of the form,

$$\int_{A_i}^{B_i} \mathrm{d}x\, g_{im}(x)[h_i(x)]^n = \delta_{mn}, \tag{5}$$

where $g_{im}(x)$ and $h_i(x)$ are some functions of x, $(i = 1, 2, \ldots, N)$, and m and n are zero or positive integers, using these identities, we can write the coherence functions of order (m, n),

$$\Gamma^{(m,n)} \equiv \mathrm{Tr}[\rho a^{\dagger m} a^n], \tag{6}$$

where $a^\dagger$ and a are the creation and annihilation operators, in the form

$$\Gamma^{(m,n)} = \int_{A_i}^{B_i} \mathrm{d}x \int_{A_j}^{B_j} \mathrm{d}y\, f_{ij}(x, y)[h_i(x)]^n[h_j^*(y)]^m, \tag{7}$$

where

$$f_{ij}(x, y) = \sum_{r,s=0}^{\infty} \Gamma^{(r,s)} g_{is}(x) g_{jr}^*(y). \tag{8}$$

Equations (6) and (7) enable us to write

$$\rho = \int_{A_i}^{B_i} \mathrm{d}x \int_{A_j}^{B_j} \mathrm{d}y\, f_{ij}(x, y) \frac{|h_i(x)\rangle\langle h_j(y)|}{\langle h_j(y)|h_i(x)\rangle}, \tag{9}$$

and give N^2 different non-diagonal representations for ρ. Let us call the representation (9) the (i, j) representation.

Some simple examples† of these identities are the following:

(i) $(A_1, B_1) = (-\infty, \infty)$, $g_{im}(x) = (m!)^{-1}(-1)^m \mathrm{e}^{\mathrm{i}m\psi}\delta^{(m)}(x)$, $h_1(x) = x\,\mathrm{e}^{-\mathrm{i}\psi}$; ψ is completely arbitrary,

(ii) $(A_2, B_2) = (0, 2\pi)$, $g_{2m}(x) = (2\pi)^{-1}a^{-m}\,\mathrm{e}^{\mathrm{i}mx}$, $h_2(x) = a\,\mathrm{e}^{-\mathrm{i}x}$; a is completely arbitrary,

(iii) $(A_3, B_3) = (-\infty, \infty)$, $g_{3m}(x) = \sum_n [(n-m)!m!]^{-1}(-1)^n(-\mathrm{i}a)^{n-m}\delta^{(n)}x$, $h_3(x) = x + \mathrm{i}a$; a is completely arbitrary,

(iv) $(A_4, B_4) = (-\infty, \infty)$, $g_{4m}(x) = \sum_n [(n-m)!m!]^{-1}\mathrm{i}^n(-a)^{n-m}\delta^{(n)}(x)$, $h_4(x) = a + \mathrm{i}x$; a is completely arbitrary.

These lead to sixteen new non-diagonal representations.

In equation (8), we expressed the weight functions in terms of coherence functions. These can also be expressed in terms of occupation number space matrix elements of ρ in the form,

$$f_{ij}(x, y) = \sum_{r,s} \rho_{rs}(r!s!)^{1/2} g_{is}(x) g_{jr}^*(y) \exp[h_i(x)h_j^*(y)]. \tag{10}$$

† It should be noted that these are not exhaustive, but, perhaps, the simplest one can think of.

Generalization of equations (9), (8) and (10) for multi-mode radiation gives

$$\rho = \prod_k \left(\int_{A_{i_k}}^{B_{i_k}} \mathrm{d}x_k \int_{A_{j_k}}^{B_{j_k}} \mathrm{d}y_k \right) f_{\{i_k\}\{j_k\}}(\{x_k\}, \{y_k\}) \frac{|\{h_{i_k}(x_k)\}\rangle\langle\{h_{j_k}(y_k)\}|}{\langle\{h_{j_k}(y)\}|\{h_{i_k}(x)\}\rangle} \tag{11}$$

$$f_{\{i_k\}\{j_k\}}(\{x_k\}, \{y_k\})$$
$$= \sum_{\{r_k\},\{s_k\}} \Gamma^{(\{r_k\},\{s_k\})} \prod_k g_{i_k s_k}(x_k) g^*_{j_k r_k}(y_k) \tag{12}$$
$$= \sum_{\{r_k\},\{s_k\}} \rho_{\{r_k\}\{s_k\}} \prod_k (r_k! s_k!)^{1/2} g_{i_k s_k}(x_k) g^*_{j_k r_k}(y_k) \exp[h_{i_k}(x_k) h^*_{j_k}(y_k)]. \tag{13}$$

Here the index k refers to the modes and the quantities $\{x_k\}$ etc stand for the sets $(x_1, x_2, \ldots)$ etc.

If both arbitrary parameters in the (1, 1) representations are taken equal to ψ, equation (9) gives

$$\rho = \int \mathrm{d}x\, \mathrm{d}y\, S(x, y)(\langle x\, \mathrm{e}^{-\mathrm{i}\psi} | y\, \mathrm{e}^{-\mathrm{i}\psi}\rangle)^{-1} |x\, \mathrm{e}^{-\mathrm{i}\psi}\rangle\langle y\, \mathrm{e}^{-\mathrm{i}\psi}|. \tag{14}$$

This has been used by Prakash *et al* (1974b, and to be reported) under the name S representation for phase-coherent and quasi-phase-coherent fields†, for which it is more convenient than the diagonal representation. As an example, consider the field having $\Gamma^{(m,n)} = \frac{1}{2}(A^m B^n + A^n B^m)$, $A \neq B$. For this field, a compact non-singular $P(\alpha)$ does not exist‡. However, if we take $\psi = 0$, $S(x, y)$ has the compact and convenient form, $\frac{1}{2}[\delta(x-A)\delta(y-B) + \delta(x-B)\delta(y-A)]$. The S representation is, thus, preferable to the diagonal representation in this case.

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† The authors (to be reported, see also 1974b) defined the phase-coherent and the quasi-phase-coherent fields by the conditions, $\Gamma^{(m,n)} = B_{m+n} \exp[\mathrm{i}(m-n)\psi]$ and $\Gamma^{(m,n)} = B_{mn} \exp[\mathrm{i}(m-n)\psi]$ respectively, where B_{m+n}, B_{mn} and ψ are real.

‡ In this case, the weight function $P(\alpha)$ is not even positive definite, as is evident from the relation,

$$\Gamma^{(0,2)} = \Gamma^{(0,2)} = \tfrac{1}{2}(A^2 + B^2) > \Gamma^{(1,1)} = AB.$$